# Technical Notes for
# *On the Economic Mechanics of Warfare*

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## Contents



*The views expressed here are mine and do not necessarily reflect the views of the Federal Reserve Bank of St. Louis or the Federal Reserve System.

# 1 Continuous time dynamic system

Consider a system $dx/dt = Ax$ where $x \in \mathbb{R}^k$ and $A$ is a $k \times k$ matrix. What is the solution?

## 1.1 Definition

Define the operator diag as

$$\operatorname{diag}\{z_1, \ldots, z_k\} = \begin{bmatrix} z_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & z_k \end{bmatrix},$$

that is a $k \times k$ matrix with the $z$s on its main diagonal and zeros elswhere. Note that

$$\begin{aligned} \kappa \operatorname{diag}\{z_1, \ldots, z_k\} &= \operatorname{diag}\{\kappa z_1, \ldots, \kappa z_k\}, \\ \sum_i \operatorname{diag}\{z_{i,1}, \ldots, z_{i,k}\} &= \operatorname{diag}\left\{\sum_i z_{i,1}, \ldots, \sum_i z_{i,k}\right\}, \\ \operatorname{diag}\{z_1, \ldots, z_k\} \operatorname{diag}\{z_1, \ldots, z_k\} &= \operatorname{diag}\left\{z_1^2, \ldots, z_k^2\right\}. \end{aligned}$$

The last line implies

$$\operatorname{diag}\{z_1, \ldots, z_k\}^n = \operatorname{diag}\{z_1^n, \ldots, z_k^n\}.$$

Define a matrix exponential as

$$\exp(tA) \equiv \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n.$$

Let $\{\lambda_1, \ldots, \lambda_k\}$ be the eigenvalues of $A$ and let $\{v_1, \ldots, v_k\}$ be the associated (column, i.e., $k \times 1$) eigenvectors. Let $P = [v_1, \ldots, v_k]$ be the $k \times k$ matrix of eigenvectors. Then,

$$\begin{aligned} \exp(tA) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} P \operatorname{diag}\{\lambda_1^n, \ldots, \lambda_k^n\} P^{-1}, \\ &= P \operatorname{diag}\left\{\sum_{n=0}^{\infty} \frac{t^n}{n!} \lambda_1^n, \ldots, \sum_{n=0}^{\infty} \frac{t^n}{n!} \lambda_k^n\right\} P^{-1}, \\ &= P \operatorname{diag}\left\{e^{t\lambda_1}, \ldots, e^{t\lambda_k}\right\} P^{-1}. \end{aligned}$$

Note that

$$\begin{aligned} \exp(0A) &= P \operatorname{diag}\left\{e^{0\lambda_1}, \ldots, e^{0\lambda_k}\right\} P^{-1}, \\ &= \operatorname{diag}\{1, \ldots, 1\}. \end{aligned}$$

## 1.2 Solution

The solution of the system $dx/dt = Ax$ is

$$x(t) = \exp(tA)\, x(0). \tag{1}$$

To see this, note first that, at $t = 0$, this equation holds: $x(0) = \exp(0A)\, x(0) = x(0)$. Second, verify that it is the solution for an arbitrary $t$. Note that

$$\begin{aligned}
\frac{dx}{dt} &= \frac{d}{dt}\sum_{n=0}^{\infty}\frac{t^n}{n!}A^n x(0) \\
&= \sum_{n=0}^{\infty}\frac{d}{dt}\left(\frac{t^n}{n!}\right)A^n x(0) \\
&= \left[\underbrace{\frac{d}{dt}\left(\frac{t^0}{0!}\right)A^0}_{0} + \sum_{n=1}^{\infty}\frac{d}{dt}\left(\frac{t^n}{n!}\right)A^n\right]x(0) \\
&= A\sum_{n=1}^{\infty}\frac{t^{n-1}}{(n-1)!}A^{n-1}x(0) \\
&= A\sum_{z=0}^{\infty}\frac{t^z}{z!}A^z x(0) \\
&= Ax
\end{aligned}$$

Thus, the solution is

$$x(t) = P\mathrm{diag}\left\{e^{t\lambda_1}, \ldots, e^{t\lambda_2}\right\}P^{-1}x(0).$$

## 1.3 Hyperbolic functions

It is useful to define the hyperbolic functions

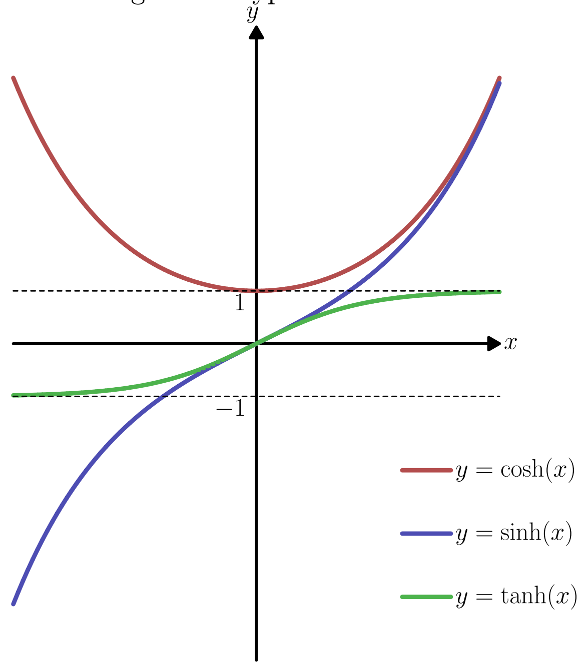
$$\begin{aligned}
\cosh(\theta) &= \frac{e^{\theta} + e^{-\theta}}{2}, \\
\sinh(\theta) &= \frac{e^{\theta} - e^{-\theta}}{2}, \\
\tanh(\theta) &= \sinh(\theta) / \cosh(\theta),
\end{aligned}$$

Figure 1 displays the hyperbolic functions. Some useful properties of the hyperbolic functions are

$$\frac{d}{d\theta}\cosh(\theta) = \sinh(x), \quad \text{and} \quad \frac{d}{d\theta}\sinh(\theta) = \cosh(x).$$

Also, $\cosh(x) \geq 1$ for all $x \in \mathbb{R}$, and $\sinh(x) < 0$ for all $x \leq 0$.

Figure 1: Hyperbolic functions




## 2 The model

The laws of motions are

$$dK_t^R/dt = -\theta^B K_t^B + X^R,$$
$$dK_t^B/dt = -\theta^R K_t^R + X^B.$$

Define the steady state as $\bar{K}^R = X^B/\theta^R$ and $\bar{K}^B = X^R/\theta^B$, and let $\tilde{K}^R = K^R - \bar{K}^R$ and $\tilde{K}^B = K^B - \bar{K}^B$. The system becomes

$$\frac{d\tilde{K}_t^R}{dt} = -\theta^B \tilde{K}_t^B,$$
$$\frac{d\tilde{K}_t^B}{dt} = -\theta^R \tilde{K}_t^R.$$

or $dx_t/dt = Mx_t$ where $x_t = \left[\tilde{K}_t^R, \tilde{K}_t^B\right]'$ and

$$M = \begin{bmatrix} 0 & -\theta^B \\ -\theta^R & 0 \end{bmatrix}.$$

The eigenvalues of $M$ solve $\lambda^2 - \theta^B\theta^R = 0$. Denote the eigenvalues by $\lambda_1$ and $\lambda_2$ with corresponding eigenvectors are $[1, v_1]'$ and $[1, v_2]'$:

$$\lambda_1 = -\sqrt{\theta^B\theta^R}, \quad \text{and} \quad \lambda_2 = \sqrt{\theta^B\theta^R},$$
$$v_1 = \sqrt{\theta^R/\theta^B}, \quad \text{and} \quad v_2 = -\sqrt{\theta^R/\theta^B}.$$

## 2.1 Dynamics

Let $P$ be the matrix of eigenvectores of $M$. Using Equation (1), the solution of $dx_t/dt = Mx_t$ is

$$\begin{aligned}
\begin{bmatrix} \tilde{K}_t^R \\ \tilde{K}_t^B \end{bmatrix} &= \frac{1}{|P|}\begin{bmatrix} 1 & 1 \\ v_1 & v_2 \end{bmatrix}\begin{bmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{bmatrix}\begin{bmatrix} v_2 & -1 \\ -v_1 & 1 \end{bmatrix}\begin{bmatrix} \tilde{K}_0^R \\ \tilde{K}_0^B \end{bmatrix} \\
&= \begin{bmatrix} e^{t\lambda_1} & e^{t\lambda_2} \\ v_1 e^{t\lambda_1} & v_2 e^{t\lambda_2} \end{bmatrix}\begin{bmatrix} \left(v_2\tilde{K}_0^R - \tilde{K}_0^B\right)/|P| \\ \left(\tilde{K}_0^B - v_1\tilde{K}_0^R\right)/|P| \end{bmatrix}
\end{aligned}$$

where $|P| = v_2 - v_1$. Thus,

$$\begin{aligned}
\tilde{K}_t^R &= \frac{e^{t\lambda_1}}{v_2 - v_1}\left(v_2\tilde{K}_0^R - \tilde{K}_0^B\right) + \frac{e^{t\lambda_2}}{v_2 - v_1}\left(\tilde{K}_0^B - v_1\tilde{K}_0^R\right) \\
\tilde{K}_t^B &= \frac{v_1 e^{t\lambda_1}}{v_2 - v_1}\left(v_2\tilde{K}_0^R - \tilde{K}_0^B\right) + \frac{v_2 e^{t\lambda_2}}{v_2 - v_1}\left(\tilde{K}_0^B - v_1\tilde{K}_0^R\right)
\end{aligned}$$

Recall that $v_2 = -v_1$. Hence, $v_2 - v_1 = -2v_1$ and the dynamic system is

$$\begin{aligned}
\tilde{K}_t^R &= \frac{e^{t\lambda_1}}{-2v_1}\left(-v_1\tilde{K}_0^R - \tilde{K}_0^B\right) + \frac{e^{t\lambda_2}}{-2v_1}\left(\tilde{K}_0^B - v_1\tilde{K}_0^R\right), \\
\tilde{K}_t^B &= \frac{v_1 e^{t\lambda_1}}{-2v_1}\left(-v_1\tilde{K}_0^R - \tilde{K}_0^B\right) + \frac{-v_1 e^{t\lambda_2}}{-2v_1}\left(\tilde{K}_0^B - v_1\tilde{K}_0^R\right),
\end{aligned}$$

or

$$\begin{aligned}
\tilde{K}_t^R &= \frac{1}{2}\left[e^{t\lambda_1}\left(\tilde{K}_0^R + \tilde{K}_0^B/v_1\right) + e^{t\lambda_2}\left(\tilde{K}_0^R - \tilde{K}_0^B/v_1\right)\right], \\
\tilde{K}_t^B &= \frac{1}{2}\left[e^{t\lambda_1}\left(\tilde{K}_0^B + v_1\tilde{K}_0^R\right) + e^{t\lambda_2}\left(\tilde{K}_0^B - v_1\tilde{K}_0^R\right)\right].
\end{aligned}$$

Rewrite this as

$$\begin{aligned}
\tilde{K}_t^R &= \frac{1}{2}\left[e^{t\lambda_1}\left(\tilde{K}_0^B + v_1\tilde{K}_0^R\right)/v_1 - e^{t\lambda_2}\left(\tilde{K}_0^B - v_1\tilde{K}_0^R\right)/v_1\right], \\
&= \frac{1}{2}\left[e^{t\lambda_1}\mathcal{A} - e^{t\lambda_2}\mathcal{B}\right]\frac{1}{v_1}, \qquad (2) \\
\tilde{K}_t^B &= \frac{1}{2}\left[e^{t\lambda_1}\left(\tilde{K}_0^B + v_1\tilde{K}_0^R\right) + e^{t\lambda_2}\left(\tilde{K}_0^B - v_1\tilde{K}_0^R\right)\right], \\
&= \frac{1}{2}\left[e^{t\lambda_1}\mathcal{A} + e^{t\lambda_2}\mathcal{B}\right]. \qquad (3)
\end{aligned}$$

where

$$\mathcal{A} = \tilde{K}_0^B + v_1\tilde{K}_0^R \quad \text{and} \quad \mathcal{B} = \tilde{K}_0^B - v_1\tilde{K}_0^R.$$

Recall that $\lambda_1 < 0$ and $\lambda_2 > 0$. The two branches of the phase diagram are $\mathcal{A} = 0$ (unstable) and $\mathcal{B} = 0$ (stable). Note that

$$\begin{aligned}
\mathcal{A}\mathcal{B} &= \left(\tilde{K}_0^B\right)^2 - \left(v_1\tilde{K}_0^R\right)^2, \qquad (4) \\
\mathcal{A} - \mathcal{B} &= 2v_1\tilde{K}_0^R, \qquad (5) \\
\mathcal{A} + \mathcal{B} &= 2\tilde{K}_0^B, \qquad (6) \\
\mathcal{A} = 0 \to \mathcal{B} &= 2\tilde{K}_0^B = -2v_1\tilde{K}_0^R, \qquad (7) \\
\mathcal{B} = 0 \to \mathcal{A} &= 2\tilde{K}_0^B = 2v_1\tilde{K}_0^R. \qquad (8)
\end{aligned}$$

Note also that

$$\begin{aligned}
4\left(v_1\tilde{K}_t^R\right)^2 &= e^{2t\lambda_1}\mathcal{A}^2 + e^{2t\lambda_2}\mathcal{B}^2 - 2\mathcal{A}\mathcal{B}, \\
4\left(\tilde{K}_t^B\right)^2 &= e^{2t\lambda_1}\mathcal{A}^2 + e^{2t\lambda_2}\mathcal{B}^2 + 2\mathcal{A}\mathcal{B}.
\end{aligned}$$

It follows that the following difference is constant at each point in time:

$$\left(\tilde{K}_t^B\right)^2 - \left(v_1\tilde{K}_t^R\right)^2 = \mathcal{A}\mathcal{B}. \qquad (9)$$

It is established from, (2) and (3), that

$$\begin{aligned}
K_t^R &= \bar{K}^R + \frac{1}{2}\left[e^{t\lambda_1}\mathcal{A} - e^{t\lambda_2}\mathcal{B}\right]\frac{1}{v_1}, \\
K_t^B &= \bar{K}^B + \frac{1}{2}\left[e^{t\lambda_1}\mathcal{A} + e^{t\lambda_2}\mathcal{B}\right],
\end{aligned}$$

where $\mathcal{A} = \tilde{K}_0^B + v_1\tilde{K}_0^R$ and $\mathcal{B} = \tilde{K}_0^B - v_1\tilde{K}_0^R$.

### 2.2 Effects of initial conditions and reinforcements

It is useful to establish the following partial derivatives

$$\frac{\partial \mathcal{A}}{\partial K_0^B} = 1, \qquad \frac{\partial \mathcal{A}}{\partial X^B} = -\frac{v_1}{\theta^R}, \qquad \frac{\partial \mathcal{A}}{\partial K_0^R} = v_1, \qquad \frac{\partial \mathcal{A}}{\partial X^R} = -\frac{1}{\theta^B},$$

and

$$\frac{\partial \mathcal{B}}{\partial K_0^B} = 1, \qquad \frac{\partial \mathcal{B}}{\partial X^B} = \frac{v_1}{\theta^R}, \qquad \frac{\partial \mathcal{B}}{\partial K_0^R} = -v_1, \qquad \frac{\partial \mathcal{B}}{\partial X^R} = -\frac{1}{\theta^B}.$$

Then,

$$\frac{\partial K_t^B}{\partial X^B} = \frac{1}{2}\left(e^{t\lambda_1}\left(-\frac{v_1}{\theta^R}\right) + e^{-t\lambda_1}\left(\frac{v_1}{\theta^R}\right)\right) = \frac{\sinh\left(t\lambda_1\right)}{\lambda_1}, \tag{10}$$

$$\frac{\partial K_t^B}{\partial X^R} = \frac{1}{\theta^B} + \frac{1}{2}\left(e^{t\lambda_1}\left(-\frac{1}{\theta^B}\right) + e^{-t\lambda_1}\left(-\frac{1}{\theta^B}\right)\right) = \frac{1-\cosh\left(t\lambda_1\right)}{\theta^B}, \tag{11}$$

$$\frac{\partial K_t^B}{\partial K_0^B} = \frac{1}{2}\left(e^{t\lambda_1}\left(1\right) + e^{-t\lambda_1}\left(1\right)\right) = \cosh\left(t\lambda_1\right), \tag{12}$$

$$\frac{\partial K_t^B}{\partial K_0^R} = \frac{1}{2}\left(e^{t\lambda_1}\left(v_1\right) + e^{-t\lambda_1}\left(-v_1\right)\right) = v_1 \sinh\left(t\lambda_1\right). \tag{13}$$

and

$$\frac{\partial K_t^R}{\partial X^B} = \frac{1}{\theta^R} + \frac{1}{2v_1}\left(e^{t\lambda_1}\left(-\frac{v_1}{\theta^B}\right) - e^{-t\lambda_1}\left(\frac{v_1}{\theta^B}\right)\right) = \frac{1-\cosh\left(t\lambda_1\right)}{\theta^R}, \tag{14}$$

$$\frac{\partial K_t^R}{\partial X^R} = \frac{1}{2v_1}\left(e^{t\lambda_1}\left(-\frac{1}{\theta^B}\right) - e^{-t\lambda_1}\left(-\frac{1}{\theta^B}\right)\right) = \frac{\sinh\left(t\lambda_1\right)}{\lambda_1}, \tag{15}$$

$$\frac{\partial K_t^R}{\partial K_0^B} = \frac{1}{2v_1}\left(e^{t\lambda_1}\left(1\right) - e^{-t\lambda_1}\left(1\right)\right) = \frac{\sinh\left(t\lambda_1\right)}{v_1}, \tag{16}$$

$$\frac{\partial K_t^R}{\partial K_0^R} = \frac{1}{2v_1}\left(e^{t\lambda_1}\left(v_1\right) - e^{-t\lambda_1}\left(-v_1\right)\right) = \cosh\left(t\lambda_1\right). \tag{17}$$

It is established, from (10)-(17), tha

$$\frac{\partial K_t^B}{\partial X^B} = \frac{\sinh(t\lambda_1)}{\lambda_1} > 0, \qquad \frac{\partial K_t^B}{\partial X^R} = \frac{1-\cosh(t\lambda_1)}{\theta^B} < 0,$$
$$\frac{\partial K_t^B}{\partial K_0^B} = \cosh(t\lambda_1) > 0, \qquad \frac{\partial K_t^B}{\partial K_0^R} = v_1 \sinh(t\lambda_1) < 0,$$

and

$$\frac{\partial K_t^R}{\partial X^B} = \frac{1-\cosh(t\lambda_1)}{\theta^R} < 0, \qquad \frac{\partial K_t^R}{\partial X^R} = \frac{\sinh(t\lambda_1)}{\lambda_1} > 0,$$
$$\frac{\partial K_t^R}{\partial K_0^B} = \frac{\sinh(t\lambda_1)}{v_1} < 0, \qquad \frac{\partial K_t^R}{\partial K_0^R} = \cosh(t\lambda_1) > 0.$$

### 2.3 Time to arbitrary value

Let $T^R(x)$ denote the date $t$ at which $K_t^R = x$:

$$2v_1\left(x - \bar{K}^R\right) = e^{T^R(x)\lambda_1}\mathcal{A} - e^{-T^R(x)\lambda_1}\mathcal{B} = z\mathcal{A} - \mathcal{B}/z,$$

where $z = \exp(T^R(x)\lambda_1)$. This is a quadratic in $z$

$$z = \frac{v_1\left(x - \bar{K}^R\right) \pm \sqrt{\left[v_1\left(x - \bar{K}^R\right)\right]^2 + \mathcal{A}\mathcal{B}}}{\mathcal{A}}.$$

Similarly, let $T^B(x)$ denote the date $t$ at which $K_t^B = x$:

$$2\left(x - \bar{K}^B\right) = e^{T^B(x)\lambda_1}\mathcal{A} + e^{-T^B(x)\lambda_1}\mathcal{B} = z\mathcal{A} + \mathcal{B}/z,$$

with solution

$$z = \frac{\left(x - \bar{K}^B\right) \pm \sqrt{\left(x - \bar{K}^B\right)^2 - \mathcal{A}\mathcal{B}}}{\mathcal{A}}.$$

The quadratic equations may have zero, one or two solutions depending on $x$. Two solutions are possible because the trajectories of $K_t^B$ and $K_t^R$ need not be monotonic. Since $\lambda_1 < 0$, a solution $z$ corresponding to a positive time must be between 0 and 1.

A remark on units — Suppose that $X^R, X^B, K_0^R$, and $K_0^B$ are multiplied by $\kappa > 0$. Then $\bar{K}^R$, $\bar{K}^B$ are also multiplied by $\kappa$. It follows that $\tilde{K}_0^R$ and $\tilde{K}_0^B$ are multiplied by $\kappa$ as well. Finally $\mathcal{A}$ and $\mathcal{B}$ are multiplied by $\kappa$. Note that, $T^R(\kappa x) = T^R(x)$ and $T^B(\kappa x) = T^B(x)$, thus the duration of war remains constant.

It is established that the time to an arbitrary weapons stock $x$ is

$$
\begin{aligned}
T^R(x) &= \frac{1}{\lambda_1}\ln\left(\frac{v_1\left(x-\bar{K}^R\right)\pm\Delta^R(x)}{\mathcal{A}}\right),\\
\Delta^R(x) &= \sqrt{\left[v_1\left(x-\bar{K}^R\right)\right]^2+\mathcal{A}\mathcal{B}},\\
T^B(x) &= \frac{1}{\lambda_1}\ln\left(\frac{\left(x-\bar{K}^B\right)\pm\Delta^B(x)}{\mathcal{A}}\right),\\
\Delta^B(x) &= \sqrt{\left(x-\bar{K}^B\right)^2-\mathcal{A}\mathcal{B}}.
\end{aligned}
$$

## 2.4 Casualties

Let $D_t^B$ and $D_t^R$ denote total casualties up to time $t$. Then

$$
\begin{aligned}
D_t^B &= \theta^R\int_0^t K_u^R du = \theta^R\int_0^t \tilde{K}_u^R du + \theta^R\int_0^t \bar{K}^R du,\\
&= \theta^R\frac{1}{2v_1}\int_0^t\left[e^{u\lambda_1}\mathcal{A}-e^{u\lambda_2}\mathcal{B}\right]du + tX^B,\\
&= -\frac{\lambda_1}{2}\left[\mathcal{A}\int_0^t e^{u\lambda_1}du-\mathcal{B}\int_0^t e^{-u\lambda_1}du\right]+tX^B,
\end{aligned}
$$

where $\int_0^t e^{u\lambda_1}du = (e^{t\lambda_1}-1)/\lambda_1$ and $\int_0^t e^{-u\lambda_1}du = -(e^{-t\lambda_1}-1)/\lambda_1$. So,

$$
\begin{aligned}
D_t^B &= -\frac{\lambda_1}{2}\left[\mathcal{A}\frac{e^{t\lambda_1}-1}{\lambda_1}+\mathcal{B}\frac{e^{-t\lambda_1}-1}{\lambda_1}\right]+tX^B,\\
&= -\frac{1}{2}\left[\mathcal{A}e^{t\lambda_1}+\mathcal{B}e^{-t\lambda_1}-2\tilde{K}_0^B\right]+tX^B,\\
&= -\frac{1}{2}\left[2\tilde{K}_t^B-2\tilde{K}_0^B\right]+tX^B,\\
&= tX^B+K_0^B-K_t^B,
\end{aligned}
\tag{18}
$$

where (6) was used. For Red casualties we have

$$
\begin{aligned}
D_t^R &= \theta^B\int_0^t K_u^B du = \theta^B\int_0^t \tilde{K}_u^B du + \theta^B\int_0^t \bar{K}^B du,,\\
&= \theta^B\frac{1}{2}\left[\mathcal{A}\int_0^t e^{u\lambda_1}du+\mathcal{B}\int_0^t e^{u\lambda_2}du\right]+tX^R,\\
&= \frac{\theta^B}{\lambda_1}\frac{1}{2}\left[\mathcal{A}e^{t\lambda_1}-\mathcal{A}-\mathcal{B}e^{-t\lambda_1}+\mathcal{B}\right]+tX^R,\\
&= \frac{\theta^B}{\lambda_1}\frac{1}{2}\left[2v_1\tilde{K}_t^R-2v_1\tilde{K}_0^R\right]+tX^R,\\
&= tX^R+K_0^R-K_t^R,
\end{aligned}
\tag{19}
$$

where (5) was used.

It is established, from (19) and (18), that

$$D_t^B = \int_0^t \theta^R K_u^R du = tX^B + K_0^B - K_t^B,$$

and

$$D_t^R = \int_0^t \theta^B K_u^B du = tX^R + K_0^R - K_t^R.$$

Casualties at time $t$ behave as

$$\begin{aligned}\frac{\partial D_t^B}{\partial X^B} &= \theta^R \int_0^t \frac{\partial K_u^R}{\partial X^B} du = \theta^R \int_0^t \frac{1-\cosh(u\lambda_1)}{\theta^R} du = t - \int_0^t \cosh(u\lambda_1)\, du, \\ &= t - \frac{\sinh(t\lambda_1)}{\lambda_1},\end{aligned} \tag{20}$$

after using (14).

$$\begin{aligned}\frac{\partial D_t^B}{\partial X^R} &= \theta^R \int_0^t \frac{\partial K_u^R}{\partial X^R} du = \theta^R \int_0^t \frac{\sinh(u\lambda_1)}{\lambda_1} du = \frac{\theta^R}{\lambda_1^2}\left(\cosh(t\lambda_1) - 1\right), \\ &= \frac{\cosh(t\lambda_1) - 1}{\theta^B},\end{aligned} \tag{21}$$

after using (15).

$$\begin{aligned}\frac{\partial D_t^B}{\partial K_0^B} &= \theta^R \int_0^t \frac{\partial K_u^R}{\partial K_0^B} du = \theta^R \int_0^t \frac{\sinh(u\lambda_1)}{v_1} du = \frac{\theta^R}{v_1\lambda_1}\left(\cosh(t\lambda_1) - 1\right), \\ &= 1 - \cosh(t\lambda_1),\end{aligned} \tag{22}$$

after using (16).

$$\begin{aligned}\frac{\partial D_t^B}{\partial K_0^R} &= \theta^R \int_0^t \frac{\partial K_u^R}{\partial K_0^R} du = \theta^R \int_0^t \cosh(u\lambda_1)\, du = \theta^R \sinh(t\lambda_1)/\lambda_1, \\ &= -v_1 \sinh(t\lambda_1).\end{aligned} \tag{23}$$

after using (17). Finally,

$$\frac{\partial D_t^R}{\partial X^B} = \theta^B \int_0^t \frac{\partial K_u^B}{\partial X^B} du = \theta^B \int_0^t \frac{\sinh(u\lambda_1)}{\lambda_1} du = \frac{\cosh(t\lambda_1) - 1}{\theta^R}, \tag{24}$$

after using (10).

$$\frac{\partial D_t^R}{\partial X^R} = \theta^B \int_0^t \frac{\partial K_u^B}{\partial X^R} du = \theta^B \int_0^t \frac{1 - \cosh(u\lambda_1)}{\theta^B} du = t - \frac{\sinh(t\lambda_1)}{\lambda_1}, \tag{25}$$

after using (11).

$$\frac{\partial D_t^R}{\partial K_0^B} = \theta^B \int_0^t \frac{\partial K_u^B}{\partial K_0^B} du = \theta^B \int_0^t \cosh(u\lambda_1)\, du = -\frac{\sinh(t\lambda_1)}{v_1}, \tag{26}$$

after using (12).

$$\frac{\partial D_t^R}{\partial K_0^R} = \theta^B \int_0^t \frac{\partial K_u^B}{\partial K_0^R} du = \theta^B v_1 \int_0^t \sinh(u\lambda_1)\, du = 1 - \cosh(t\lambda_1), \tag{27}$$

ater using (13).

It is established, from (20)-(27), that

$$\frac{\partial D_t^B}{\partial X^B} = t - \frac{\sinh(t\lambda_1)}{\lambda_1} < 0, \qquad \frac{\partial D_t^B}{\partial X^R} = \frac{\cosh(t\lambda_1) - 1}{\theta^B} > 0,$$
$$\frac{\partial D_t^B}{\partial K_0^B} = 1 - \cosh(t\lambda_1) < 0, \qquad \frac{\partial D_t^B}{\partial K_0^R} = -v_1 \sinh(t\lambda_1) > 0,$$

and

$$\frac{\partial D_t^R}{\partial X^B} = \frac{\cosh(t\lambda_1) - 1}{\theta^R} > 0, \qquad \frac{\partial D_t^R}{\partial X^R} = t - \frac{\sinh(t\lambda_1)}{\lambda_1} < 0,$$
$$\frac{\partial D_t^R}{\partial K_0^B} = -\frac{\sinh(t\lambda_1)}{v_1} > 0, \qquad \frac{\partial D_t^R}{\partial K_0^R} = 1 - \cosh(t\lambda_1) < 0.$$

Note that

$$\frac{\partial D_t^B}{\partial K_0^B} = 1 - \frac{\partial K_t^B}{\partial K_0^B} \quad \text{and} \quad \frac{\partial D_t^B}{\partial X^B} = t - \frac{\partial K_t^B}{\partial X^B},$$

and

$$\frac{\partial D_t^R}{\partial K_0^R} = 1 - \frac{\partial K_t^R}{\partial K_0^R} \quad \text{and} \quad \frac{\partial D_t^R}{\partial X^R} = t - \frac{\partial K_t^R}{\partial X^R}.$$

On the other hand

$$\frac{\partial D_t^B}{\partial K_0^R} = -\frac{\partial K_t^B}{\partial K_0^R} \quad \text{and} \quad \frac{\partial D_t^B}{\partial X^R} = -\frac{\partial K_t^B}{\partial X^R},$$

and

$$\frac{\partial D_t^R}{\partial K_0^B} = -\frac{\partial K_t^R}{\partial K_0^B} \quad \text{and} \quad \frac{\partial D_t^R}{\partial X^B} = -\frac{\partial K_t^R}{\partial X^B}.$$

### 2.5 Military conclusion

Consider a military victory for Blue at $\tau$:

$$\mathcal{B} > 0 \quad \text{and} \quad K_\tau^R = 0. \tag{28}$$

#### 2.5.1 Final Blue weapons stock

Equation (9) implies

$$K_\tau^B = \bar{K}^B + \underbrace{\sqrt{\left(v_1 \bar{K}^R\right)^2 + \mathcal{A}\mathcal{B}}}_{\Delta}.$$

where

$$\begin{aligned}
\Delta^2 &= \left(v_1 \bar{K}^R\right)^2 + \mathcal{A}\mathcal{B}, \\
&= \left(v_1 \bar{K}^R\right)^2 + \left(\tilde{K}_0^B\right)^2 - \left(v_1 \tilde{K}_0^R\right)^2, \\
&= \left(v_1 \bar{K}^R\right)^2 + \left(\tilde{K}_0^B\right)^2 - v_1^2 \left(K_0^R - \bar{K}^R\right)^2, \\
&= \left(v_1 \bar{K}^R\right)^2 + \left(\tilde{K}_0^B\right)^2 - v_1^2 \left(\left(K_0^R\right)^2 + \left(\bar{K}^R\right)^2 - 2K_0^R \bar{K}^R\right), \\
&= \left(K_0^B - \bar{K}^B\right)^2 - \left(v_1 K_0^R\right)^2 + v_1^2 2 K_0^R \bar{K}^R.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{\partial K_\tau^B}{\partial X^B} &= \frac{1}{2}\left(\left(v_1 \bar{K}^R\right)^2 + \mathcal{A}\mathcal{B}\right)^{1/2-1} \frac{\partial}{\partial X^B}\left(\left(v_1 \bar{K}^R\right)^2 + \mathcal{A}\mathcal{B}\right), \\
&= \frac{1}{2}\left(\Delta^2\right)^{-1/2} v_1^2 2 K_0^R / \theta^R, \\
&= \frac{1}{\theta^B} \frac{K_0^R}{K_\tau^B - \bar{K}^B}.
\end{aligned} \tag{29}$$

$$\begin{aligned}
\frac{\partial K_\tau^B}{\partial X^R} &= \frac{1}{\theta^B} + \frac{1}{2}\left(\left(v_1 \bar{K}^R\right)^2 + \mathcal{A}\mathcal{B}\right)^{1/2-1} \frac{\partial}{\partial X^R}\left(\left(v_1 \bar{K}^R\right)^2 + \mathcal{A}\mathcal{B}\right), \\
&= \frac{1}{\theta^B} - \frac{1}{2}\left(\Delta^2\right)^{-1/2} 2\left(K_0^B - \bar{K}^B\right)\frac{1}{\theta^B}, \\
&= \frac{1}{\theta^B} \frac{K_\tau^B - K_0^B}{K_\tau^B - \bar{K}^B}.
\end{aligned} \tag{30}$$

$$
\begin{aligned}
\frac{\partial K_\tau^B}{\partial K_0^B} &= \frac{1}{2}\left(\left(v_1\bar{K}^R\right)^2 + \mathcal{AB}\right)^{1/2-1}\frac{\partial}{\partial K_0^B}\left(\left(v_1\bar{K}^R\right)^2 + \mathcal{AB}\right),\\
&= \frac{1}{2}\left(\Delta^2\right)^{-1/2} 2\left(K_0^B - \bar{K}^B\right),\\
&= \frac{K_0^B - \bar{K}^B}{K_\tau^B - \bar{K}^B}.
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
\frac{\partial K_\tau^B}{\partial K_0^R} &= \frac{1}{2}\left(\left(v_1\bar{K}^R\right)^2 + \mathcal{AB}\right)^{1/2-1}\frac{\partial}{\partial K_0^R}\left(\left(v_1\bar{K}^R\right)^2 + \mathcal{AB}\right),\\
&= \frac{1}{2}\left(\Delta^2\right)^{-1/2}\left(-v_1^2 2K_0^R + 2v_1^2\bar{K}^R\right),\\
&= -v_1^2\frac{K_0^R - \bar{K}^R}{K_\tau^B - \bar{K}^B}.
\end{aligned}
\tag{32}
$$

Assume (28) is satisfied. Then, it is established, from (29)-(32), that

$$
\frac{\partial K_\tau^B}{\partial X^B} = \frac{1}{\theta^B}\frac{K_0^R}{K_\tau^B - \bar{K}^B}, \qquad \frac{\partial K_\tau^B}{\partial X^R} = \frac{1}{\theta^B}\frac{K_\tau^B - K_0^B}{K_\tau^B - \bar{K}^B}
$$

and

$$
\frac{\partial K_\tau^B}{\partial K_0^B} = \frac{K_0^B - \bar{K}^B}{K_\tau^B - \bar{K}^B}, \qquad \frac{\partial K_\tau^B}{\partial K_0^R} = -v_1^2\frac{K_0^R - \bar{K}^R}{K_\tau^B - \bar{K}^B}
$$

Note that $K_\tau^B > \bar{K}^B$. Thus, $K_\tau^B > \bar{K}_0^B$ whenever $\bar{K}^B > K_0^B$. The interesting case is $\bar{K}^B < K_0^B$, or $K_0^B - \bar{K}^B > 0$. Then

$$
\begin{aligned}
K_\tau^B &\geq K_0^B,\\
\sqrt{\left(v_1\bar{K}^R\right)^2 + \mathcal{AB}} &\geq K_0^B - \bar{K}^B,\\
\left(K_0^B - \bar{K}^B\right)^2 - \left(v_1K_0^R\right)^2 + v_1^2 2K_0^R\bar{K}^R &\geq \left(K_0^B - \bar{K}^B\right)^2,\\
2\bar{K}^R &\geq K_0^R.
\end{aligned}
$$

Assume (28) is satisfied. Then, it is established that

$$
K_\tau^B > \bar{K}^B
$$

and

$$
K_\tau^B \geq K_0^B \quad \text{if and only if} \quad K_0^R \leq 2\bar{K}^R.
$$

### 2.5.2 Duration

At the end of the war, $K_\tau^R = 0$. Taking derivatives yields

$$\frac{\partial K_\tau^R}{\partial \tau}d\tau + \frac{\partial K_\tau^R}{\partial X^B}dX^B = 0 \Rightarrow \frac{\partial \tau}{\partial X^B} = -\frac{\partial K_\tau^R/\partial X^B}{\partial K_\tau^R/\partial \tau} = \frac{\left(1-\cosh\left(\tau\lambda_1\right)\right)/\theta^R}{\theta^B \tilde{K}_\tau^B}, \tag{33}$$

$$\frac{\partial K_\tau^R}{\partial \tau}d\tau + \frac{\partial K_\tau^R}{\partial X^B}dX^R = 0 \Rightarrow \frac{\partial \tau}{\partial X^R} = -\frac{\partial K_\tau^R/\partial X^R}{\partial K_\tau^R/\partial \tau} = \frac{\sinh\left(\tau\lambda_1\right)/\lambda_1}{\theta^B \tilde{K}_\tau^B}, \tag{34}$$

$$\frac{\partial K_\tau^R}{\partial \tau}d\tau + \frac{\partial K_\tau^R}{\partial X^B}dK_0^B = 0 \Rightarrow \frac{\partial \tau}{\partial K_0^B} = -\frac{\partial K_\tau^R/\partial K_0^B}{\partial K_\tau^R/\partial \tau} = \frac{\sinh\left(\tau\lambda_1\right)/v_1}{\theta^B \tilde{K}_\tau^B}, \tag{35}$$

$$\frac{\partial K_\tau^R}{\partial \tau}d\tau + \frac{\partial K_\tau^R}{\partial X^B}dK_0^R = 0 \Rightarrow \frac{\partial \tau}{\partial K_0^R} = -\frac{\partial K_\tau^R/\partial K_0^R}{\partial K_\tau^R/\partial \tau} = \frac{\cosh\left(\tau\lambda_1\right)}{\theta^B \tilde{K}_\tau^B}, \tag{36}$$

where (14)-(17) and $\partial K_\tau^R/\partial\tau = X^R - \theta^B K_\tau^B = \theta^B\left(\bar{K}^B - K_\tau^B\right) = -\theta^B \tilde{K}_\tau^B < 0$ were used.

Assume (28) is satisfied. Then, it is established, from (33)-(36), that

$$\frac{\partial \tau}{\partial X^B} = \frac{\left(1-\cosh\left(\tau\lambda_1\right)\right)/\theta^R}{\theta^B \tilde{K}_\tau^B} < 0 \qquad \frac{\partial \tau}{\partial X^R} = \frac{\sinh\left(\tau\lambda_1\right)/\lambda_1}{\theta^B \tilde{K}_\tau^B} > 0,$$

and

$$\frac{\partial \tau}{\partial K_0^B} = \frac{\sinh\left(\tau\lambda_1\right)/v_1}{\theta^B \tilde{K}_\tau^B} < 0 \qquad \frac{\partial \tau}{\partial K_0^R} = \frac{\cosh\left(\tau\lambda_1\right)}{\theta^B \tilde{K}_\tau^B} > 0.$$

The level curves of $\tau$ in the $\left(K_0^R, K_0^B\right)$ plane are defined by

$$d\tau = 0 = \frac{\partial \tau}{\partial K_0^R}dK_0^R + \frac{\partial \tau}{\partial K_0^B}dK_0^B \Rightarrow \left.\frac{dK_0^B}{dK_0^R}\right|_{d\tau=0} = -\frac{\partial\tau/\partial K_0^R}{\partial\tau/\partial K_0^B}$$

so that

$$\left.\frac{dK_0^B}{dK_0^R}\right|_{d\tau=0} = -v_1 \underbrace{\frac{\cosh\left(\tau\lambda_1\right)}{\sinh\left(\tau\lambda_1\right)}}_{<-1} > 0.$$

The level curves of $\tau$ in the $\left(X^R, X^B\right)$ plane are defined by

$$d\tau = 0 = \frac{\partial \tau}{\partial X^R}dX^R + \frac{\partial \tau}{\partial X^B}dX^B \Rightarrow \left.\frac{dX^B}{dX^R}\right|_{d\tau=0} = -\frac{\partial\tau/\partial X^R}{\partial\tau/\partial X^B}$$

so that

$$\left.\frac{dX^B}{dX^R}\right|_{d\tau=0} = -\frac{\sinh\left(\tau\lambda_1\right)/\lambda_1}{\left(1-\cosh\left(\tau\lambda_1\right)\right)/\theta^R} = v_1 \underbrace{\frac{\sinh\left(\tau\lambda_1\right)}{1-\cosh\left(\tau\lambda_1\right)}}_{>1} > 0.$$

#### 2.5.3 Casualties

Final Blue casualties depend upon resources:

$$\begin{aligned}\frac{\partial D_\tau^B}{\partial X^B} &= \theta^R \int_0^\tau \frac{\partial K_u^R}{\partial X^B} du = \theta^R \int_0^\tau \frac{1-\cosh(u\lambda_1)}{\theta^R} du = \tau - \int_0^\tau \cosh(u\lambda_1)\, du, \\ &= \tau - \frac{\sinh(\tau\lambda_1)}{\lambda_1},\end{aligned} \tag{37}$$

after using (14).

$$\begin{aligned}\frac{\partial D_\tau^B}{\partial X^R} &= \theta^R \int_0^\tau \frac{\partial K_u^R}{\partial X^R} du = \theta^R \int_0^\tau \frac{\sinh(u\lambda_1)}{\lambda_1} du = \frac{\theta^R}{\lambda_1^2}\left(\cosh(\tau\lambda_1) - 1\right), \\ &= \frac{\cosh(\tau\lambda_1) - 1}{\theta^B},\end{aligned} \tag{38}$$

after using (15).

$$\begin{aligned}\frac{\partial D_\tau^B}{\partial K_0^B} &= \theta^R \int_0^\tau \frac{\partial K_u^R}{\partial K_0^B} du = \theta^R \int_0^\tau \frac{\sinh(u\lambda_1)}{v_1} du = \frac{\theta^R}{v_1\lambda_1}\left(\cosh(\tau\lambda_1) - 1\right), \\ &= 1 - \cosh(\tau\lambda_1),\end{aligned} \tag{39}$$

after using (16). Finally,

$$\begin{aligned}\frac{\partial D_\tau^B}{\partial K_0^R} &= \theta^R \int_0^\tau \frac{\partial K_u^R}{\partial K_0^R} du = \theta^R \int_0^\tau \cosh(u\lambda_1)\, du = \theta^R \sinh(\tau\lambda_1)/\lambda_1, \\ &= -v_1 \sinh(\tau\lambda_1),\end{aligned} \tag{40}$$

after using (17). Finally, Red casualties depends on resources as well:

$$\begin{aligned}\frac{\partial D_\tau^R}{\partial X^B} &= \theta^B \int_0^\tau \frac{\partial K_u^B}{\partial X^B} du = \theta^B K_\tau^B \frac{\partial \tau}{\partial X^B} + \theta^B \int_0^\tau \frac{\partial K_u^B}{\partial X^B} du, \\ &= \theta^B K_\tau^B \frac{1-\cosh(\tau\lambda_1)}{\theta^R\theta^B \tilde{K}_\tau^B} + \theta^B \int_0^\tau \frac{\sinh(u\lambda_1)}{\lambda_1} du, \\ &= \frac{K_\tau^B}{\tilde{K}_\tau^B}\frac{1-\cosh(\tau\lambda_1)}{\theta^R} + \frac{\cosh(\tau\lambda_1) - 1}{\theta^R} \\ &= \frac{1-\cosh(\tau\lambda_1)}{\theta^R}\left(\frac{\bar{K}^B}{\tilde{K}_\tau^B}\right),\end{aligned} \tag{41}$$

after using (10) and (33).

$$\frac{\partial D_\tau^R}{\partial X^R} = \theta^B \int_0^\tau \frac{\partial K_u^B du}{\partial X^R} = \theta^B K_\tau^B \frac{\partial \tau}{\partial X^R} + \theta^B \int_0^\tau \frac{\partial K_u^B}{\partial X^R} du,$$
$$= \theta^B K_\tau^B \frac{\sinh(\tau\lambda_1)}{\theta^B \tilde{K}_\tau^B \lambda_1} + \theta^B \int_0^\tau \frac{1-\cosh(u\lambda_1)}{\theta^B} du,$$
$$= \frac{K_\tau^B}{\tilde{K}_\tau^B} \frac{\sinh(\tau\lambda_1)}{\lambda_1} + \tau - \frac{\sinh(\tau\lambda_1)}{\lambda_1}$$
$$= \tau + \frac{\sinh(\tau\lambda_1)}{\lambda_1} \left( \frac{\bar{K}^B}{\tilde{K}_\tau^B} \right), \tag{42}$$

after using (11) and (34).

$$\frac{\partial D_\tau^R}{\partial K_0^B} = \theta^B \int_0^\tau \frac{\partial K_u^B du}{\partial K_0^B} = \theta^B K_\tau^B \frac{\partial \tau}{\partial K_0^B} + \theta^B \int_0^\tau \frac{\partial K_u^B}{\partial K_0^B} du,$$
$$= \theta^B K_\tau^B \frac{\sinh(\tau\lambda_1)}{\theta^B \tilde{K}_\tau^B v_1} + \theta^B \int_0^\tau \cosh(u\lambda_1) du,$$
$$= \frac{K_\tau^B}{\tilde{K}_\tau^B} \frac{\sinh(\tau\lambda_1)}{v_1} + \frac{\theta^B}{\lambda_1} \sinh(\tau\lambda_1)$$
$$= \frac{\sinh(\tau\lambda_1)}{v_1} \left( \frac{\bar{K}^B}{\tilde{K}_\tau^B} \right), \tag{43}$$

after using (12) and (35). Finally,

$$\frac{\partial D_\tau^R}{\partial K_0^R} = \theta^B \int_0^\tau \frac{\partial K_u^B du}{\partial K_0^R} = \theta^B K_\tau^B \frac{\partial \tau}{\partial K_0^R} + \theta^B \int_0^\tau \frac{\partial K_u^B}{\partial K_0^R} du,$$
$$= \theta^B K_\tau^B \frac{\cosh(\tau\lambda_1)}{\theta^B \tilde{K}_\tau^B} + \theta^B v_1 \int_0^\tau \sinh(u\lambda_1) du,$$
$$= \frac{K_\tau^B}{\tilde{K}_\tau^B} \cosh(\tau\lambda_1) + \frac{\theta^B v_1}{\lambda_1} (\cosh(\tau\lambda_1) - 1)$$
$$= 1 + \cosh(\tau\lambda_1) \left( \frac{\bar{K}^B}{\tilde{K}_\tau^B} \right), \tag{44}$$

after using (13) and (36).

Assume (28) is satisfied. Then, it is established, from (37)-(44), that

$$\frac{\partial D^B_\tau}{\partial X^B} = \tau - \frac{\sinh(\tau\lambda_1)}{\lambda_1} < 0, \qquad \frac{\partial D^B_\tau}{\partial X^R} = \frac{\cosh(\tau\lambda_1) - 1}{\theta^B} > 0,$$

$$\frac{\partial D^B_\tau}{\partial K^B_0} = 1 - \cosh(\tau\lambda_1) < 0, \qquad \frac{\partial D^B_\tau}{\partial K^R_0} = -v_1 \sinh(\tau\lambda_1) > 0,$$

and

$$\frac{\partial D^R_\tau}{\partial X^B} = \frac{1 - \cosh(\tau\lambda_1)}{\theta^R}\left(\frac{\bar{K}^B}{\tilde{K}^B_\tau}\right) < 0, \qquad \frac{\partial D^R_\tau}{\partial X^R} = \tau + \frac{\sinh(\tau\lambda_1)}{\lambda_1}\left(\frac{\bar{K}^B}{\tilde{K}^B_\tau}\right) > 0,$$

$$\frac{\partial D^R_\tau}{\partial K^B_0} = \frac{\sinh(\tau\lambda_1)}{v_1}\left(\frac{\bar{K}^B}{\tilde{K}^B_\tau}\right) < 0, \qquad \frac{\partial D^R_\tau}{\partial K^R_0} = 1 + \cosh(\tau\lambda_1)\left(\frac{\bar{K}^B}{\tilde{K}^B_\tau}\right) > 0.$$

### 2.6 Political conclusion

Let $\bar{D}^B$ and $\bar{D}^R$ denote the threshold levels of casualties for Blue and Red, respectively. The war concludes when

$$D^B_t = \bar{D}^B \quad \text{or} \quad D^R_t = \bar{D}^R,$$

whichever comes first (if it is not a military conclusion).

#### 2.6.1 Duration

Let $\tau^B$ and $\tau^R$ denote the date at which the thresholds are reached. Using (18) and (19), it follows that

$$\begin{aligned} K^B_{\tau^B} &= \tau^B X^B + K^B_0 - \bar{D}^B, \\ K^R_{\tau^R} &= \tau^R X^R + K^R_0 - \bar{D}^R. \end{aligned}$$

Implicitly differentiating the first equation yields

$$\begin{aligned} \left(\frac{\partial K^B_{\tau^B}}{\partial \tau^B} - X^B\right) d\tau^B + \frac{\partial K^B_{\tau^B}}{\partial X^B} dX^B = \tau^B dX^B, \\ \frac{d\tau^B}{dX^B} &= \frac{\tau^B - \partial K^B_{\tau^B}/\partial X^B}{\partial K^B_{\tau^B}/\partial \tau^B - X^B}, \\ &= \frac{1}{\theta^R K^R_{\tau^B}}\left(\frac{\sinh(\tau^B\lambda_1)}{\lambda_1} - \tau^B\right), \end{aligned} \tag{45}$$

where the last line uses (10).

$$\left(\frac{\partial K^B_{\tau^B}}{\partial \tau^B} - X^B\right) d\tau^B + \frac{\partial K^B_{\tau^B}}{\partial X^R} dX^R = 0,$$

$$\frac{d\tau^B}{dX^R} = -\frac{\partial K^B_{\tau^B}/\partial X^R}{\partial K^B_{\tau^B}/\partial \tau^B - X^B},$$

$$= \frac{1}{\theta^R K^R_{\tau^B}} \left(\frac{1-\cosh\left(\tau^B \lambda_1\right)}{\theta^B}\right), \tag{46}$$

where the last line uses (11).

$$\left(\frac{\partial K^B_{\tau^B}}{\partial \tau^B} - X^B\right) d\tau^B + \frac{\partial K^B_{\tau^B}}{\partial K^B_0} dK^B_0 = dK^B_0,$$

$$\frac{d\tau^B}{dK^B_0} = \frac{1-\partial K^B_{\tau^B}/\partial K^B_0}{\partial K^B_{\tau^B}/\partial \tau^B - X^B},$$

$$= \frac{1}{\theta^R K^R_{\tau^B}} \left(\cosh\left(\tau^B \lambda_1\right) - 1\right), \tag{47}$$

where the last line uses (12). Finally,

$$\left(\frac{\partial K^B_{\tau^B}}{\partial \tau^B} - X^B\right) d\tau^B + \frac{\partial K^B_{\tau^B}}{\partial K^R_0} dK^R_0 = 0,$$

$$\frac{d\tau^B}{dK^R_0} = -\frac{\partial K^B_{\tau^B}/\partial K^R_0}{\partial K^B_{\tau^B}/\partial \tau^B - X^B},$$

$$= \frac{1}{\theta^R K^R_{\tau^B}} v_1 \sinh\left(\tau^B \lambda_1\right), \tag{48}$$

where the last line follows from (13).

It is established, from (45)-(48), that

$$\frac{\partial \tau^B}{\partial X^B} = \frac{1}{\theta^R K^R_{\tau^B}} \left(\frac{\sinh\left(\tau^B \lambda_1\right)}{\lambda_1} - \tau^B\right) > 0, \qquad \frac{\partial \tau^B}{\partial X^R} = \frac{1}{\theta^R K^R_{\tau^B}} \left(\frac{1-\cosh\left(\tau^B \lambda_1\right)}{\theta^B}\right) < 0,$$

and

$$\frac{\partial \tau^B}{\partial K^B_0} = \frac{1}{\theta^R K^R_{\tau^B}} \left(\cosh\left(\tau^B \lambda_1\right) - 1\right) > 0, \qquad \frac{\partial \tau^B}{\partial K^R_0} = \frac{1}{\theta^R K^R_{\tau^B}} v_1 \sinh\left(\tau^B \lambda_1\right) < 0.$$

Proceeding with Red,

$$\left(\frac{\partial K^R_{\tau^R}}{\partial \tau^R} - X^R\right) d\tau^R + \frac{\partial K^R_{\tau^R}}{\partial X^B} dX^B = 0,$$

$$\begin{aligned} \frac{d\tau^R}{dX^B} &= -\frac{\partial K^R_{\tau^R}/\partial X^B}{\partial K^R_{\tau^R}/\partial \tau^R - X^R} \\ &= \frac{1}{\theta^B K^B_{\tau^R}} \left(\frac{1-\cosh\left(\tau^R \lambda_1\right)}{\theta^R}\right), \end{aligned} \tag{49}$$

where the last line uses (14).

$$\left(\frac{\partial K^R_{\tau^R}}{\partial \tau^R} - X^R\right) d\tau^R + \frac{\partial K^R_{\tau^R}}{\partial X^R} dX^R = \tau^R dX^R,$$

$$\begin{aligned} \frac{d\tau^R}{dX^R} &= \frac{\tau^R - \partial K^R_{\tau^R}/\partial X^R}{\partial K^R_{\tau^R}/\partial \tau^R - X^R}, \\ &= \frac{1}{\theta^B K^B_{\tau^R}} \left(\frac{\sinh\left(\tau^R \lambda_1\right)}{\lambda_1} - \tau^R\right), \end{aligned} \tag{50}$$

where the last line uses (15).

$$\left(\frac{\partial K^R_{\tau^R}}{\partial \tau^R} - X^R\right) d\tau^R + \frac{\partial K^R_{\tau^R}}{\partial K^B_0} dK^B_0 = 0,$$

$$\begin{aligned} \frac{d\tau^R}{dK^B_0} &= -\frac{\partial K^R_{\tau^R}/\partial K^B_0}{\partial K^R_{\tau^R}/\partial \tau^R - X^R}, \\ &= \frac{1}{\theta^B K^B_{\tau^R}} \left(\frac{\sinh\left(\tau^R \lambda_1\right)}{v_1}\right), \end{aligned} \tag{51}$$

where the last line uses (16). Finally,

$$\left(\frac{\partial K^R_{\tau^R}}{\partial \tau^R} - X^R\right) d\tau^R + \frac{\partial K^R_{\tau^R}}{\partial K^R_0} dK^R_0 = dK^R_0,$$

$$\begin{aligned} \frac{d\tau^R}{dK^R_0} &= \frac{1 - \partial K^R_{\tau^R}/\partial K^R_0}{\partial K^R_{\tau^R}/\partial \tau^R - X^R}, \\ &= \frac{1}{\theta^B K^B_{\tau^R}} \left(\cosh\left(\tau^R \lambda_1\right) - 1\right), \end{aligned} \tag{52}$$

where the last line uses (17).

It is established, from (49)-(52), that

$$\frac{\partial \tau^R}{\partial X^B} = \frac{1}{\theta^B K^B_{\tau^R}} \left( \frac{1-\cosh\left(\tau^R \lambda_1\right)}{\theta^R} \right) < 0, \qquad \frac{\partial \tau^R}{\partial X^R} = \frac{1}{\theta^B K^B_{\tau^R}} \left( \frac{\sinh\left(\tau^R \lambda_1\right)}{\lambda_1} - \tau^R \right) > 0,$$

and

$$\frac{\partial \tau^R}{\partial K^B_0} = \frac{1}{\theta^B K^B_{\tau^R}} \left( \frac{\sinh\left(\tau^R \lambda_1\right)}{v_1} \right) < 0, \qquad \frac{\partial \tau^R}{\partial K^R_0} = \frac{1}{\theta^B K^B_{\tau^R}} \left(\cosh\left(\tau^R \lambda_1\right) - 1\right) > 0.$$

The level curves of $\tau^B$ in the $\left(K^R_0, K^B_0\right)$ plane are defined by

$$d\tau^B = 0 = \frac{\partial \tau^B}{\partial K^B_0} dK^B_0 + \frac{\partial \tau^B}{\partial K^R_0} dK^R_0 \Rightarrow \left.\frac{dK^B_0}{dK^R_0}\right|_{d\tau^B=0} = -\frac{\partial \tau^B / \partial K^R_0}{\partial \tau^B / \partial K^B_0},$$

so that

$$\left.\frac{dK^B_0}{dK^R_0}\right|_{d\tau^B=0} = -v_1 \underbrace{\frac{\sinh\left(\tau^B \lambda_1\right)}{\cosh\left(\tau^B \lambda_1\right) - 1}}_{<-1}.$$

In the $\left(X^R, X^B\right)$ plane, the level curves have slopes

$$\left.\frac{dX^B}{dX^R}\right|_{d\tau^B=0} = -\frac{\frac{1-\cosh\left(\tau^B \lambda_1\right)}{\theta^B}}{\frac{\sinh\left(\tau^B \lambda_1\right)}{\lambda_1} - \tau^B} = v_1 \underbrace{\frac{1-\cosh\left(\tau^B \lambda_1\right)}{\sinh\left(\tau^B \lambda_1\right) - \tau^B \lambda_1}}_{>1}.$$

Comparing the slopes of the level curves for $\tau$ and $\tau^B$:

$$\left.\frac{dK^B_0}{dK^R_0}\right|_{d\tau^B=0} = -v_1 \underbrace{\frac{\sinh\left(\tau^B \lambda_1\right)}{\cosh\left(\tau^B \lambda_1\right) - 1}}_{<-1} \quad \text{and} \quad \left.\frac{dK^B_0}{dK^R_0}\right|_{d\tau=0} = -v_1 \underbrace{\frac{\cosh\left(\tau \lambda_1\right)}{\sinh\left(\tau \lambda_1\right)}}_{<-1} > 0.$$

Note that

$$-1 > \frac{\cosh\left(\tau \lambda_1\right)}{\sinh\left(\tau \lambda_1\right)} > \frac{\sinh\left(\tau^B \lambda_1\right)}{\cosh\left(\tau^B \lambda_1\right) - 1},$$

so that

$$\left.\frac{dK^B_0}{dK^R_0}\right|_{d\tau=0} < \left.\frac{dK^B_0}{dK^R_0}\right|_{d\tau^B=0}.$$

Following an increase in $K^R_0$ by $dK^R_0$. The increase in $K^B_0$ that maintains $\tau$ constant is too little to maintain $\tau^B$ constant, and so $\tau^B$ decreases.

### 2.6.2 Red casualties when Red initiates a political conclusion

These derivatives should be zero by assumptions since, when Red initiates a political conclusion its casualties must be $\bar{D}^R$ by definition. Verifying these derivatives are zero is a check on the model's solution:

$$\begin{aligned}
\frac{\partial D^R_{\tau^R}}{\partial X^B} &= \frac{\partial}{\partial X^B}\left(\theta^B \int_0^{\tau^R} K^B_u du\right) = \theta^B K^B_{\tau^R}\frac{\partial \tau^R}{\partial X^B} + \theta^B \int_0^{\tau^R} \frac{\partial}{\partial X^B} K^B_u du, \\
&= \theta^B K^B_{\tau^R}\frac{1}{\theta^B K^B_{\tau^R}}\left(\frac{1-\cosh\left(\tau^R\lambda_1\right)}{\theta^R}\right) + \theta^B \int_0^{\tau^R} \frac{\sinh\left(u\lambda_1\right)}{\lambda_1} du, \\
&= \frac{1-\cosh\left(\tau^R\lambda_1\right)}{\theta^R} + \frac{\theta^B}{\lambda_1^2}\left(\cosh\left(\tau^R\lambda_1\right)-1\right), \\
&= \frac{1-\cosh\left(\tau^R\lambda_1\right)}{\theta^R} + \frac{\cosh\left(\tau^R\lambda_1\right)-1}{\theta^R} = 0,
\end{aligned}$$

where (49) and (10) are used.

$$\begin{aligned}
\frac{\partial D^R_{\tau^R}}{\partial X^R} &= \frac{\partial}{\partial X^R}\left(\theta^B \int_0^{\tau^R} K^B_u du\right) = \theta^B K^B_{\tau^R}\frac{\partial \tau^R}{\partial X^R} + \theta^B \int_0^{\tau^R} \frac{\partial}{\partial X^R} K^B_u du, \\
&= \theta^B K^B_{\tau^R}\frac{1}{\theta^B K^B_{\tau^R}}\left(\frac{\sinh\left(\tau^R\lambda_1\right)}{\lambda_1} - \tau^R\right) + \theta^B \int_0^{\tau^R} \frac{1-\cosh\left(u\lambda_1\right)}{\theta^B} du, \\
&= \frac{\sinh\left(\tau^R\lambda_1\right)}{\lambda_1} - \tau^R + \int_0^{\tau^R}\left(1-\cosh\left(u\lambda_1\right)\right) du, \\
&= \frac{\sinh\left(\tau^R\lambda_1\right)}{\lambda_1} - \tau^R + \tau^R - \frac{\sinh\left(\tau^R\lambda_1\right)}{\lambda_1} = 0,
\end{aligned}$$

where (50) and (11) are used.

$$\begin{aligned}
\frac{\partial D^R_{\tau^R}}{\partial K^B_0} &= \frac{\partial}{\partial K^B_0}\left(\theta^B \int_0^{\tau^R} K^B_u du\right) = \theta^B K^B_{\tau^R}\frac{\partial \tau^R}{\partial K^B_0} + \theta^B \int_0^{\tau^R} \frac{\partial}{\partial K^B_0} K^B_u du, \\
&= \theta^B K^B_{\tau^R}\frac{1}{\theta^B K^B_{\tau^R}}\left(\frac{\sinh\left(\tau^R\lambda_1\right)}{v_1}\right) + \theta^B \int_0^{\tau^R} \cosh\left(u\lambda_1\right) du, \\
&= \frac{\sinh\left(\tau^R\lambda_1\right)}{v_1} + \frac{\theta^B}{\lambda_1}\sinh\left(\tau^R\lambda_1\right) = 0,
\end{aligned}$$

where (51) and (12) are used.

$$
\begin{aligned}
\frac{\partial D^R_{\tau^R}}{\partial K^R_0} &= \frac{\partial}{\partial K^R_0}\left(\theta^B \int_0^{\tau^R} K^B_u du\right) = \theta^B K^B_{\tau^R}\frac{\partial \tau^R}{\partial K^R_0} + \theta^B \int_0^{\tau^R} \frac{\partial}{\partial K^R_0} K^B_u du,\\
&= \theta^B K^B_{\tau^R}\frac{1}{\theta^B K^B_{\tau^R}}\left(\cosh\left(\tau^R\lambda_1\right) - 1\right) + \theta^B \int_0^{\tau^R} v_1 \sinh\left(u\lambda_1\right) du,\\
&= \cosh\left(\tau^R\lambda_1\right) - 1 + \frac{v_1\theta^B}{\lambda_1}\left(\cosh\left(\tau^R\lambda_1\right) - 1\right) = 0,
\end{aligned}
$$

where (52) and (13) are used.

### 2.6.3 Blue casualties when Red initiates a political conclusion

$$
\begin{aligned}
\frac{\partial D^B_{\tau^R}}{\partial X^B} &= \frac{\partial}{\partial X^B}\left(\theta^R \int_0^{\tau^R} K^R_u du\right) = \theta^R K^R_{\tau^R}\frac{\partial \tau^R}{\partial X^B} + \theta^R \int_0^{\tau^R} \frac{\partial}{\partial X^B} K^R_u du,\\
&= \theta^R K^R_{\tau^R}\frac{1}{\theta^B K^B_{\tau^R}}\frac{1-\cosh\left(\tau^R\lambda_1\right)}{\theta^R} + \theta^R \int_0^{\tau^R} \frac{1-\cosh\left(u\lambda_1\right)}{\theta^R} du,\\
&= \theta^R K^R_{\tau^R}\frac{1}{\theta^B K^B_{\tau^R}}\frac{1-\cosh\left(\tau^R\lambda_1\right)}{\theta^R} + \int_0^{\tau^R} \left(1-\cosh\left(u\lambda_1\right)\right) du,\\
&= \frac{\theta^R K^R_{\tau^R}}{\theta^B K^B_{\tau^R}}\frac{1-\cosh\left(\tau^R\lambda_1\right)}{\theta^R} + \int_0^{\tau^R} 1 du - \int_0^{\tau^R} \cosh\left(u\lambda_1\right) du,\\
&= \underbrace{\frac{\theta^R K^R_{\tau^R}}{\theta^B K^B_{\tau^R}}\frac{1-\cosh\left(\tau^R\lambda_1\right)}{\theta^R}}_{<0} + \underbrace{\tau^R - \frac{\sinh\left(\tau^R\lambda_1\right)}{\lambda_1}}_{<0} < 0,
\end{aligned}
$$

where (49) and (14) are used.

$$
\begin{aligned}
\frac{\partial D^B_{\tau^R}}{\partial X^R} &= \frac{\partial}{\partial X^R}\left(\theta^R \int_0^{\tau^R} K^R_u du\right) = \theta^R K^R_{\tau^R}\frac{\partial \tau^R}{\partial X^R} + \theta^R \int_0^{\tau^R} \frac{\partial}{\partial X^R} K^R_u du,\\
&= \frac{\theta^R K^R_{\tau^R}}{\theta^B K^B_{\tau^R}}\left(\frac{\sinh\left(\tau^R\lambda_1\right)}{\lambda_1} - \tau^R\right) + \theta^R \int_0^{\tau^R} \frac{\sinh\left(u\lambda_1\right)}{\lambda_1} du,\\
&= \frac{\theta^R K^R_{\tau^R}}{\theta^B K^B_{\tau^R}}\left(\frac{\sinh\left(\tau^R\lambda_1\right)}{\lambda_1} - \tau^R\right) + \frac{\theta^R}{\lambda_1}\left(\int_0^{\tau^R} \sinh\left(u\lambda_1\right) du\right),\\
&= \underbrace{\frac{\theta^R K^R_{\tau^R}}{\theta^B K^B_{\tau^R}}\left(\frac{\sinh\left(\tau^R\lambda_1\right)}{\lambda_1} - \tau^R\right)}_{>0} + \underbrace{\frac{1}{\theta^B}\left(\cosh\left(\tau^R\lambda_1\right) - 1\right)}_{>0} > 0,
\end{aligned}
$$

where (50) and (15) are used.

$$
\begin{aligned}
\frac{\partial D^B_{\tau^R}}{\partial K^B_0} &= \frac{\partial}{\partial K^B_0}\left(\theta^R \int_0^{\tau^R} K^R_u du\right) = \theta^R K^R_{\tau^R} \frac{\partial \tau^R}{\partial K^B_0} + \theta^R \int_0^{\tau^R} \frac{\partial}{\partial K^B_0} K^R_u du, \\
&= \theta^R K^R_{\tau^R} \frac{1}{\theta^B K^B_{\tau^R}} \frac{\sinh\left(\tau^R \lambda_1\right)}{v_1} + \theta^R \int_0^{\tau^R} \frac{\sinh\left(u\lambda_1\right)}{v_1} du, \\
&= \frac{\theta^R K^R_{\tau^R}}{\theta^B K^B_{\tau^R}} \frac{\sinh\left(\tau^R \lambda_1\right)}{v_1} + \frac{\theta^R}{v_1} \int_0^{\tau^R} \sinh\left(u\lambda_1\right) du, \\
&= \frac{\theta^R K^R_{\tau^R}}{\theta^B K^B_{\tau^R}} \frac{\sinh\left(\tau^R \lambda_1\right)}{v_1} + \frac{\theta^R}{v_1 \lambda_1} \left(\cosh\left(\tau^R \lambda_1\right) - 1\right), \\
&= \underbrace{\frac{\theta^R K^R_{\tau^R}}{\theta^B K^B_{\tau^R}} \frac{\sinh\left(\tau^R \lambda_1\right)}{v_1}}_{<0} + \underbrace{1 - \cosh\left(\tau^R \lambda_1\right)}_{<0} < 0,
\end{aligned}
$$

where (51) and (16) are used.

$$
\begin{aligned}
\frac{\partial D^B_{\tau^R}}{\partial K^R_0} &= \frac{\partial}{\partial K^R_0}\left(\theta^R \int_0^{\tau^R} K^R_u du\right) = \theta^R K^R_{\tau^R} \frac{\partial \tau^R}{\partial K^R_0} + \theta^R \int_0^{\tau^R} \frac{\partial}{\partial K^R_0} K^R_u du, \\
&= \frac{\theta^R K^R_{\tau^R}}{\theta^B K^B_{\tau^R}} \left(\cosh\left(\tau^R \lambda_1\right) - 1\right) + \theta^R \int_0^{\tau^R} \cosh\left(u\lambda_1\right) du, \\
&= \underbrace{\frac{\theta^R K^R_{\tau^R}}{\theta^B K^B_{\tau^R}} \left(\cosh\left(\tau^R \lambda_1\right) - 1\right)}_{>0} + \underbrace{\frac{\theta^R}{\lambda_1} \sinh\left(\tau^R \lambda_1\right)}_{>0} > 0,
\end{aligned}
$$

where (52) and (17) are used.

#### 2.6.4 Blue casualties when Blue initiate a political conclusion

These derivatives must be zero by construction.

$$
\begin{aligned}
\frac{\partial D^B_{\tau^B}}{\partial X^B} &= \frac{\partial}{\partial X^B}\left(\theta^R \int_0^{\tau^B} K^R_u du\right) = \theta^R K^R_{\tau^B} \frac{\partial \tau^B}{\partial X^B} + \theta^R \int_0^{\tau^B} \frac{\partial}{\partial X^B} K^R_u du, \\
&= \theta^R K^R_{\tau^B} \frac{1}{\theta^R K^R_{\tau^B}} \left(\frac{\sinh\left(\tau^B \lambda_1\right)}{\lambda_1} - \tau^B\right) + \theta^R \int_0^{\tau^B} \left(\frac{1 - \cosh\left(u\lambda_1\right)}{\theta^R}\right) du, \\
&= \frac{\sinh\left(\tau^B \lambda_1\right)}{\lambda_1} - \tau^B + \int_0^{\tau^B} \left(1 - \cosh\left(u\lambda_1\right)\right) du, \\
&= \frac{\sinh\left(\tau^B \lambda_1\right)}{\lambda_1} - \tau^B + \tau^B - \int_0^{\tau^B} \cosh\left(u\lambda_1\right) du, \\
&= \frac{\sinh\left(\tau^B \lambda_1\right)}{\lambda_1} - \tau^B + \tau^B - \frac{\sinh\left(\tau^B \lambda_1\right)}{\lambda_1} = 0,
\end{aligned}
$$

where (45) and (14) are used.

$$\frac{\partial D_{\tau^B}^B}{\partial X^R} = \frac{\partial}{\partial X^R}\left(\theta^R \int_0^{\tau^B} K_u^R du\right) = \theta^R K_{\tau^B}^R \frac{\partial \tau^B}{\partial X^R} + \theta^R \int_0^{\tau^B} \frac{\partial}{\partial X^R} K_u^R du,$$

$$= \theta^R K_{\tau^B}^R \frac{1}{\theta^R K_{\tau^B}^R}\left(\frac{1-\cosh\left(\tau^B \lambda_1\right)}{\theta^B}\right) + \theta^R \int_0^{\tau^B} \frac{\sinh\left(u\lambda_1\right)}{\lambda_1} du,$$

$$= \frac{1-\cosh\left(\tau^B\lambda_1\right)}{\theta^B} + \frac{\theta^R}{\lambda_1^2}\left(\cosh\left(\tau^B\lambda_1\right)-1\right),$$

$$= \frac{1-\cosh\left(\tau^B\lambda_1\right)}{\theta^B} + \frac{1}{\theta^B}\left(\cosh\left(\tau^B\lambda_1\right)-1\right) = 0,$$

where (46) and (15) are used.

$$\frac{\partial D_{\tau^B}^B}{\partial K_0^B} = \frac{\partial}{\partial K_0^B}\left(\theta^R \int_0^{\tau^B} K_u^R du\right) = \theta^R K_{\tau^B}^R \frac{\partial \tau^B}{\partial K_0^B} + \theta^R \int_0^{\tau^B} \frac{\partial}{\partial K_0^B} K_u^R du,$$

$$= \theta^R K_{\tau^B}^R \frac{1}{\theta^R K_{\tau^B}^R}\left(\cosh\left(\tau^B\lambda_1\right)-1\right) + \theta^R \int_0^{\tau^B} \frac{\sinh\left(u\lambda_1\right)}{v_1} du,$$

$$= \cosh\left(\tau^B\lambda_1\right) - 1 + \frac{\theta^R}{v_1\lambda_1}\left(\cosh\left(\tau^B\lambda_1\right)-1\right) = 0,$$

where (47) and (16) are used.

$$\frac{\partial D_{\tau^B}^B}{\partial K_0^R} = \frac{\partial}{\partial K_0^R}\left(\theta^R \int_0^{\tau^B} K_u^R du\right) = \theta^R K_{\tau^B}^R \frac{\partial \tau^B}{\partial K_0^R} + \theta^R \int_0^{\tau^B} \frac{\partial}{\partial K_0^R} K_u^R du$$

$$= \theta^R K_{\tau^B}^R \frac{1}{\theta^R K_{\tau^B}^R} v_1 \sinh\left(\tau^B\lambda_1\right) + \theta^R \int_0^{\tau^B} \cosh\left(u\lambda_1\right) du,$$

$$= v_1 \sinh\left(\tau^B\lambda_1\right) + \frac{\theta^R}{\lambda_1}\sinh\left(\tau^B\lambda_1\right),$$

$$= v_1 \sinh\left(\tau^B\lambda_1\right) - v_1 \sinh\left(\tau^B\lambda_1\right) = 0,$$

where (48) and (17) are used.

#### 2.6.5 Red casualties when Blue initiate a political conclusion

$$
\begin{aligned}
\frac{\partial D^R_{\tau^B}}{\partial X^B} &= \frac{\partial}{\partial X^B}\left(\theta^B \int_0^{\tau^B} K^B_u du\right) = \theta^B K^B_{\tau^B}\frac{\partial \tau^B}{\partial X^B} + \theta^B \int_0^{\tau^B} \frac{\partial}{\partial X^B} K^B_u du, \\
&= \theta^B K^B_{\tau^B}\frac{1}{\theta^R K^R_{\tau^B}}\left(\frac{\sinh\left(\tau^B\lambda_1\right)}{\lambda_1} - \tau^B\right) + \theta^B \int_0^{\tau^B} \frac{\sinh\left(u\lambda_1\right)}{\lambda_1} du, \\
&= \underbrace{\frac{\theta^B K^B_{\tau^B}}{\theta^R K^R_{\tau^B}}\left(\frac{\sinh\left(\tau^B\lambda_1\right)}{\lambda_1} - \tau^B\right)}_{>0} + \underbrace{\frac{\cosh\left(\tau^B\lambda_1\right) - 1}{\theta^R}}_{>0} > 0,
\end{aligned}
$$

where (45) and (10) are used.

$$
\begin{aligned}
\frac{\partial D^R_{\tau^B}}{\partial X^R} &= \frac{\partial}{\partial X^R}\left(\theta^B \int_0^{\tau^B} K^B_u du\right) = \theta^B K^B_{\tau^B}\frac{\partial \tau^B}{\partial X^R} + \theta^B \int_0^{\tau^B} \frac{\partial}{\partial X^R} K^B_u du, \\
&= \theta^B K^B_{\tau^B}\frac{1}{\theta^R K^R_{\tau^B}}\left(\frac{1 - \cosh\left(\tau^B\lambda_1\right)}{\theta^B}\right) + \theta^B \int_0^{\tau^B} \frac{1 - \cosh\left(u\lambda_1\right)}{\theta^B} du, \\
&= \frac{\theta^B K^B_{\tau^B}}{\theta^R K^R_{\tau^B}}\left(\frac{1 - \cosh\left(\tau^B\lambda_1\right)}{\theta^B}\right) + \int_0^{\tau^B} \left(1 - \cosh\left(u\lambda_1\right)\right) du, \\
&= \underbrace{\frac{\theta^B K^B_{\tau^B}}{\theta^R K^R_{\tau^B}}\left(\frac{1 - \cosh\left(\tau^B\lambda_1\right)}{\theta^B}\right)}_{<0} + \underbrace{\tau^B - \frac{\sinh\left(\tau^B\lambda_1\right)}{\lambda_1}}_{<0} < 0,
\end{aligned}
$$

where (46) and (11) are used.

$$
\begin{aligned}
\frac{\partial D^R_{\tau^B}}{\partial K^B_0} &= \frac{\partial}{\partial K^B_0}\left(\theta^B \int_0^{\tau^B} K^B_u du\right) = \theta^B K^B_{\tau^B}\frac{\partial \tau^B}{\partial K^B_0} + \theta^B \int_0^{\tau^B} \frac{\partial}{\partial K^B_0} K^B_u du, \\
&= \theta^B K^B_{\tau^B}\frac{1}{\theta^R K^R_{\tau^B}}\left(\cosh\left(\tau^B\lambda_1\right) - 1\right) + \theta^B \int_0^{\tau^B} \cosh\left(u\lambda_1\right) du, \\
&= \underbrace{\frac{\theta^B K^B_{\tau^B}}{\theta^R K^R_{\tau^B}}\left(\cosh\left(\tau^B\lambda_1\right) - 1\right)}_{>0} - \underbrace{\frac{\sinh\left(\tau^B\lambda_1\right)}{v_1}}_{<0} > 0,
\end{aligned}
$$

where (47) and (12) are used.

$$\begin{aligned}
\frac{\partial D^R_{\tau^B}}{\partial K^R_0} &= \frac{\partial}{\partial K^R_0}\left(\theta^B \int_0^{\tau^B} K^B_u du\right) = \theta^B K^B_{\tau^B} \frac{\partial \tau^B}{\partial K^R_0} + \theta^B \int_0^{\tau^B} \frac{\partial}{\partial K^R_0} K^B_u du, \\
&= \theta^B K^B_{\tau^B} \frac{1}{\theta^R K^R_{\tau^B}} v_1 \sinh\left(\tau^B \lambda_1\right) + \theta^B \int_0^{\tau^B} v_1 \sinh\left(u\lambda_1\right) du, \\
&= \frac{\theta^B K^B_{\tau^B}}{\theta^R K^R_{\tau^B}} v_1 \sinh\left(\tau^B \lambda_1\right) + \frac{\theta^B v_1}{\lambda_1}\left(\cosh\left(\tau^B \lambda_1\right) - 1\right), \\
&= \underbrace{\frac{\theta^B K^B_{\tau^B}}{\theta^R K^R_{\tau^B}} v_1 \sinh\left(\tau^B \lambda_1\right)}_{<0} + \underbrace{1 - \cosh\left(\tau^B \lambda_1\right)}_{<0} < 0,
\end{aligned}$$

where (48) and (13) are used.